



Meshless Methods

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Element Free Galerkin Method (EFG):

1. Developed by Belytschko (1994) based on DEM.
2. MLS approximation is used for the construction of the shape function.
3. Galerkin weak form is employed to develop the discretized system equation.
4. Cells of the background mesh for integration are required to carry out the integration required to calculate system matrices.
5. EFG is conforming due to:
 1. The use of MLS shape functions that are consistent and compatible.
 2. The use of the constrained Galerkin approach to impose the essential boundary condition.

EFG formulation:

- Assume a typical solid mechanics problem,

$$\begin{cases} L^T \sigma + b = 0 & \text{on } \Omega \\ u = \bar{u} & \text{on } \Gamma_u \\ \sigma n = \bar{t} & \text{on } \Gamma_t \end{cases}$$

- σ = the stress tensor
- u = the displacement vector
- b = the body force vector
- $\bar{t}$ = the prescribed traction on the natural (stress) boundary.
- $\bar{u}$ = the prescribed displacement on the essential (displacement) boundary.
- n = the vector of unit normal on the natural boundary.

EFG formulation:

$$\begin{cases} L^T \sigma + b = 0 & \text{on } \Omega \\ u = \bar{u} & \text{on } \Gamma_u \\ \sigma n = \bar{t} & \text{on } \Gamma_t \end{cases}$$

$$L = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \\ \partial/\partial y & \partial/\partial x & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix}$$

EFG Formulation:

- The MLS approximation is consistent and compatible; so the Galerkin procedure can then be used to establish a set of discretized system equations for the displacement parameters (different from the actual displacement at nodes)
- The Galerkin weak form can be given by:

$$\int_{\Omega} \delta(Lu)^T (CLu) d\Omega - \int_{\Omega} \delta u^T b d\Omega - \int_{\Gamma_t} \delta u^T \bar{t} d\Gamma = 0$$

- Using $\varepsilon = Lu$ and $\sigma = C\varepsilon$

$$\int_{\Omega} \delta \varepsilon^T \sigma d\Omega - \int_{\Omega} \delta u^T b d\Omega - \int_{\Gamma_t} \delta u^T \bar{t} d\Gamma = 0$$

EFG Formulation:

- The constrained Galerkin weak form:

$$\begin{aligned} & \int_{\Omega} \delta(Lu)^T (CLu) d\Omega - \int_{\Omega} \delta u^T b d\Omega - \int_{\Gamma_t} \delta u^T \bar{t} d\Gamma \\ & - \int_{\Gamma_u} \delta \lambda^T (u - \bar{u}) d\Gamma - \int_{\Gamma_u} \delta u^T \lambda d\Gamma = 0 \end{aligned}$$

- Using $\varepsilon = Lu$ and $\sigma = C\varepsilon$

$$\begin{aligned} & \int_{\Omega} \delta \varepsilon^T \sigma d\Omega - \int_{\Omega} \delta u^T b d\Omega - \int_{\Gamma_t} \delta u^T \bar{t} d\Gamma \\ & - \int_{\Gamma_u} \delta \lambda^T (u - \bar{u}) d\Gamma - \int_{\Gamma_u} \delta u^T \lambda d\Gamma = 0 \end{aligned}$$

EFG Formulation:

- The MLS approximation is now used to express the displacements at any point of interest x using nodes in the support domain of the point x .

$$\mathbf{u}^h = \begin{Bmatrix} \mathbf{u}^h \\ \mathbf{v}^h \end{Bmatrix} = \begin{Bmatrix} \mathbf{u} \\ \mathbf{v} \end{Bmatrix}^h$$

$$u^h(x) = \sum_{I=1}^n \phi_I(x) u_I$$

$$v^h(x) = \sum_{I=1}^n \phi_I(x) v_I$$

EFG Formulation:

$$\mathbf{u}^h = \begin{Bmatrix} \mathbf{u} \\ \mathbf{v} \end{Bmatrix}^h = \sum_{I=1}^n \begin{bmatrix} \phi_I & 0 \\ 0 & \phi_I \end{bmatrix} \begin{Bmatrix} u_I \\ v_I \end{Bmatrix} = \sum_{I=1}^n \Phi_I u_I$$

ϕ_I = MLS shape function

Φ_I = Matrix of shape functions

EFG Formulation:

- To calculate the strains:

$$\varepsilon = Lu^h = L \sum_{I=1}^n \Phi_I u_I$$

$$\varepsilon = \sum_{I=1}^n \begin{bmatrix} \partial / \partial x & 0 \\ 0 & \partial / \partial y \\ \partial / \partial y & \partial / \partial x \end{bmatrix} \begin{bmatrix} \phi_I & 0 \\ 0 & \phi_I \end{bmatrix} u_I$$

$$\varepsilon = \sum_{I=1}^n \begin{bmatrix} \phi_{I,x} & 0 \\ 0 & \phi_{I,y} \\ \phi_{I,y} & \phi_{I,x} \end{bmatrix} u_I = \sum_{I=1}^n B_I u_I$$

EFG Formulation:

- The Lagrange multiplier λ is an unknown function of the coordinates, which needs to be interpolated using the nodes on the essential boundaries to obtain a discretized set of system equations:

$$\lambda(x) = \sum_{I=1}^{n_\lambda} N_I(s) \lambda_I, \quad x \in \Gamma_u$$

n_λ = Number of nodes used for this interpolation

s = The arc-length along the essential boundary

λ_I = The Lagrange multiplier at node I on the essential boundary

$N_I(s)$ = A Lagrange interpolant used in conventional FEM

EFG Formulation:

- The Lagrange interpolant of order n has the general form of:

$$N_k^n(s) = \frac{(s - s_0)(s - s_1) \cdots (s - s_{k-1})(s - s_{k+1}) \cdots (s - s_n)}{(s_k - s_0)(s_k - s_1) \cdots (s_k - s_{k-1})(s_k - s_{k+1}) \cdots (s_k - s_n)}$$

- For a first order Lagrange interpolant:

$$N_0(s) = \frac{s - s_1}{s_0 - s_1} \quad \text{Lagrange interpolant at point } s=s_0$$

$$N_1(s) = \frac{s - s_0}{s_1 - s_0} \quad \text{Lagrange interpolant at point } s=s_1$$

- In this case, the Lagrange multiplier at s is interpolated using two nodes that are located before and after s . For higher order interpolants, more nodes on the boundary have to be used.

EFG Formulation:

- The variation of Lagrange multiplier $\delta\lambda$ can be obtained:

$$\lambda(x) = \sum_{I=1}^{n_\lambda} N_I(s) \lambda_I, \quad x \in \Gamma_u$$

$$\delta\lambda(x) = \sum_{I=1}^{n_\lambda} N_I(s) \delta\lambda_I, \quad x \in \Gamma_u$$

EFG weak Formulation:

$$\begin{aligned} & \int_{\Omega} \delta(Lu)^T (CLu) d\Omega - \int_{\Omega} \delta u^T b d\Omega - \int_{\Gamma_t} \delta u^T \bar{t} d\Gamma \\ & - \int_{\Gamma_u} \delta \lambda^T (u - \bar{u}) d\Gamma - \int_{\Gamma_u} \delta u^T \lambda d\Gamma = 0 \end{aligned}$$

$$\begin{aligned} & \int_{\Omega} \delta \varepsilon^T \sigma d\Omega - \int_{\Omega} \delta u^T b d\Omega - \int_{\Gamma_t} \delta u^T \bar{t} d\Gamma \\ & - \int_{\Gamma_u} \delta \lambda^T (u - \bar{u}) d\Gamma - \int_{\Gamma_u} \delta u^T \lambda d\Gamma = 0 \end{aligned}$$

EFG weak Formulation:

$$W_k - W_b - W_t - W_{\lambda_1} - W_{\lambda_2} = 0$$

$$W_k = \int_{\Omega} \delta \left(\sum_{I=1}^n B_I u_I \right)^T (C \sum_{J=1}^n B_J u_J) d\Omega$$

$$W_b = \int_{\Omega} \delta \left(\sum_{I=1}^n \Phi_I u_I \right)^T b d\Omega$$

$$W_t = \int_{\Gamma_t} \delta \left(\sum_{I=1}^n \Phi_I u_I \right)^T \bar{t} d\Gamma$$

$$W_{\lambda_1} = \int_{\Gamma_u} \delta \lambda^T \left(\left(\sum_{I=1}^n \Phi_I u_I \right) - \bar{u} \right) d\Gamma$$

$$W_{\lambda_2} = \int_{\Gamma_u} \delta \left(\sum_{I=1}^n \Phi_I u_I \right)^T \lambda d\Gamma$$

EFG weak Formulation (W_k):

$$W_k = \int_{\Omega} \delta \left(\sum_{I=1}^n B_I u_I \right)^T (C \sum_{J=1}^n B_J u_J) d\Omega$$

$$W_k = \int_{\Omega} \delta \left(\sum_{I=1}^n u_I^T B_I^T \right) (C \sum_{J=1}^n B_J u_J) d\Omega$$

- The summation, variation and integration are linear exchangeable operators:

$$W_k = \sum_{I=1}^n \sum_{J=1}^n \delta u_I^T \left[\int_{\Omega} B_I^T C B_J d\Omega \right] u_J$$

$$W_k = \sum_{I=1}^{n_t} \sum_{J=1}^{n_t} \delta u_I^T K_{IJ} u_J \quad (n \rightarrow n_t)$$

$$K_{IJ} = \int_{\Omega} B_I^T C B_J d\Omega$$

EFG weak Formulation(W_k):

$$W_k = \sum_{I=1}^{n_t} \sum_{J=1}^{n_t} \delta u_I^T K_{IJ} u_J$$

$$\begin{aligned} W_k = & \delta u_1^T K_{11} u_1 + \delta u_1^T K_{12} u_2 + \cdots + \delta u_1^T K_{1n_t} u_{n_t} + \\ & \delta u_2^T K_{21} u_1 + \delta u_2^T K_{22} u_2 + \cdots + \delta u_2^T K_{2n_t} u_{n_t} + \\ & \vdots \\ & \delta u_{n_t}^T K_{n_t 1} u_1 + \delta u_{n_t}^T K_{n_t 2} u_2 + \cdots + \delta u_{n_t}^T K_{n_t n_t} u_{n_t} \end{aligned}$$

EFG weak Formulation(W_k):

$$W_k = \delta U^T K U$$

$$K = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n_t} \\ K_{21} & K_{22} & \cdots & K_{2n_t} \\ \vdots & \vdots & & \vdots \\ K_{n_t1} & K_{n_t2} & \cdots & K_{n_tn_t} \end{bmatrix}_{(2n_t \times 2n_t)} = \text{The global stiffness matrix}$$

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n_t} \end{bmatrix}_{(2n_t \times 1)} = \text{The global displacement parameter vector}$$

$$u_I = \begin{Bmatrix} u_I \\ v_I \end{Bmatrix}$$

EFG weak Formulation(W_k):

1. K_{IJ} is a 2×2 nodal stiffness matrix.
2. The integration is over the entire domain and all the nodes can be involved (the summation limits have to be changed to n_t).
3. The nodal stiffness matrix K_{IJ} contains the stiffness coefficients between nodes I and J evaluated at a point in the problem domain. It is a function of coordinates and needs to be integrated over the entire problem domain.
4. K_{IJ} has to be assembled to the global stiffness matrix K , as long as the nodes I and J are covered by the support domain of at least one quadrature point. If nodes I and J are far apart and if they do not share the same support domain of any quadrature point, K_{IJ} vanishes and thus there is no need to compute.

EFG weak Formulation(W_k):

5. As long as the support domain is compact and does not cover too wide a problem domain, many K_{IJ} will be zero, and K will be sparse.
6. If the nodes are properly numbered, K will also be banded.
7. K is symmetric (because of symmetry of C)

$$K_{IJ}^T = \int_{\Omega} [B_I^T C B_J]^T d\Omega = \int_{\Omega} [B_J^T C^T B_I] d\Omega = \int_{\Omega} [B_J^T C B_I] d\Omega = K_{JI}$$

EFG weak Formulation (W_b):

$$W_b = \int_{\Omega} \delta \left(\sum_{I=1}^n \Phi_I u_I \right)^T b d\Omega$$

$$W_b = \sum_{I=1}^n \delta u_I^T \int_{\Omega} \Phi_I^T b d\Omega$$

$$W_b = \sum_{I=1}^{n_t} \delta u_I^T f_I \quad (n \rightarrow n_t)$$

$$f_I = \int_{\Omega} \Phi_I^T b d\Omega \quad = \text{the nodal force vector}$$

$$f_I = \begin{Bmatrix} f_{xI} \\ f_{yI} \end{Bmatrix}$$

EFG weak Formulation(W_b):

- The information is over the entire domain and all the nodes can be involved (the summation limits have to be changed to n_t).

$$W_b = \sum_{I=1}^{n_t} \delta u_I^T f_I = \delta u_1^T f_1 + \delta u_2^T f_2 + \cdots + \delta u_{n_t}^T f_{n_t}$$

$$W_b = \delta U^T F$$

$$F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n_t} \end{bmatrix}_{(2n_t)} = \text{the global force vector}$$

EFG weak Formulation (W_t):

$$W_t = \int_{\Gamma_t} \delta \left(\sum_{I=1}^n \Phi_I u_I \right)^T \bar{t} d\Gamma$$

$$W_t = \sum_{I=1}^n \delta u_I^T \int_{\Gamma_t} \Phi_I^T \bar{t} d\Gamma$$

$$W_t = \sum_{I=1}^{n_{\Gamma_t}} \delta u_I^T f_I \quad (n \rightarrow n_{\Gamma_t})$$

$$f_I = \int_{\Gamma_t} \Phi_I^T \bar{t} d\Gamma \quad = \text{the nodal force vector}$$

EFG weak Formulation(W_t):

$$W_t = \sum_{I=1}^{n_{\Gamma_t}} \delta u_I^T f_I = \delta u_1^T f_1 + \delta u_2^T f_2 + \cdots + \delta u_{n_{\Gamma_t}}^T f_{n_{\Gamma_t}}$$

$$W_t = \delta U^T F$$

F = the global force vector

1. contribution from the external body force
2. contribution from the external force applied on the natural boundaries.

EFG weak Formulation (W_{λ_1}):

$$W_{\lambda_1} = \int_{\Gamma_u} \delta \lambda^T \left(\left(\sum_{I=1}^n \Phi_I u_I \right) - \bar{u} \right) d\Gamma$$

$$W_{\lambda_1} = \int_{\Gamma_u} \delta \left(\sum_{I=1}^{n_\lambda} N_I \lambda_I \right)^T \sum_{J=1}^n \Phi_J u_J d\Gamma - \int_{\Gamma_u} \delta \left(\sum_{I=1}^{n_\lambda} N_I \lambda_I \right)^T \bar{u} d\Gamma$$

$$W_{\lambda_1} = \sum_{I=1}^{n_\lambda} \sum_{J=1}^n \delta \lambda_I^T \int_{\Gamma_u} N_I^T \Phi_J d\Gamma u_J - \sum_{I=1}^{n_\lambda} \delta \lambda_I^T \int_{\Gamma_u} N_I^T \bar{u} d\Gamma$$

$$G_{IJ}^T = - \int_{\Gamma_u} N_I^T \Phi_J d\Gamma$$

$$q_I = - \int_{\Gamma_u} N_I^T \bar{u} d\Gamma$$

EFG weak Formulation (W_{λ_1}):

$$W_{\lambda_1} = - \sum_{I=1}^{n_{\lambda_t}} \sum_{J=1}^{n_t} \delta \lambda_I^T G_{IJ}^T u_J + \sum_{I=1}^{n_{\lambda_t}} \delta \lambda_I^T q_I$$

$$W_{\lambda_1} = -\delta \lambda^T G^T U + \delta \lambda^T q$$

1. n_{λ_t} is the total number of nodes on the essential boundary.
2. G_{IJ} is a 2×2 matrix.
3. G is a global $(2n_t \times 2n_{\lambda_t})$ matrix formed by assembling its nodal matrix G_{IJ} .
4. q ($2n_{\lambda_t}$) is assembled using its nodal vector q_I .

EFG weak Formulation (W_{λ_2}):

$$W_{\lambda_2} = \int_{\Gamma_u} \delta \left(\sum_{I=1}^n \Phi_I u_I \right)^T \lambda d\Gamma$$

$$W_{\lambda_2} = \int_{\Gamma_u} \delta \left(\sum_{I=1}^n \Phi_I u_I \right)^T \left(\sum_{J=1}^{n_\lambda} N_J \lambda_J \right)^T d\Gamma$$

$$W_{\lambda_2} = \sum_{I=1}^n \sum_{J=1}^{n_\lambda} \delta u_I^T \int_{\Gamma_u} \Phi_I^T N_J d\Gamma \lambda_J$$

$$G_{IJ} = - \int_{\Gamma_u} \Phi_I^T N_J d\Gamma$$

EFG weak Formulation (W_{λ_1}):

$$W_{\lambda_2} = - \sum_{I=1}^{n_t} \sum_{J=1}^{n_{\lambda_t}} \delta u_I^T G_{IJ} \lambda_J$$

$$W_{\lambda_2} = -\delta U^T G \lambda$$

EFG weak Formulation:

$$W_k - W_b - W_t - W_{\lambda_1} - W_{\lambda_2} = 0$$

$$W_k = \int_{\Omega} \delta \left(\sum_{I=1}^n B_I u_I \right)^T (C \sum_{J=1}^n B_J u_J) d\Omega$$

$$W_b = \int_{\Omega} \delta \left(\sum_{I=1}^n \Phi_I u_I \right)^T b d\Omega$$

$$W_t = \int_{\Gamma_t} \delta \left(\sum_{I=1}^n \Phi_I u_I \right)^T \bar{t} d\Gamma$$

$$W_{\lambda_1} = \int_{\Gamma_u} \delta \lambda^T \left(\left(\sum_{I=1}^n \Phi_I u_I \right) - \bar{u} \right) d\Gamma$$

$$W_{\lambda_2} = \int_{\Gamma_u} \delta \left(\sum_{I=1}^n \Phi_I u_I \right)^T \lambda d\Gamma$$

EFG weak Formulation:

$$W_k - W_b - W_t - W_{\lambda_1} - W_{\lambda_2} = 0$$

$$W_k = \delta U^T K U$$

$$W_b = \delta U^T F$$

$$W_t = \delta U^T F$$

$$W_{\lambda_1} = -\delta \lambda^T G^T U + \delta \lambda^T q$$

$$W_{\lambda_2} = -\delta U^T G \lambda$$

EFG weak Formulation:

$$W_k - W_b - W_t - W_{\lambda_1} - W_{\lambda_2} = 0$$

$$\delta U^T K U - \delta U^T F - \delta \lambda^T (G^T U - q) - \delta U^T G \lambda = 0$$

$$\delta U^T [K U + G \lambda - F] + \delta \lambda^T [G^T U - q] = 0$$

For arbitrary variations of δU , $\delta \lambda$

$$\begin{cases} K U + G \lambda - F = 0 \\ G^T U - q = 0 \end{cases}$$

EFG weak Formulation:

$$\begin{bmatrix} K & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} U \\ \lambda \end{bmatrix} = \begin{bmatrix} F \\ q \end{bmatrix}$$

$$K_{IJ} = \int_{\Omega} B_I^T C B_J d\Omega \quad \Phi_I = \begin{bmatrix} \phi_I & 0 \\ 0 & \phi_I \end{bmatrix}$$

$$G_{IJ}^T = - \int_{\Gamma_u} N_I^T \Phi_J d\Gamma \quad B_I = L \Phi_I = \begin{bmatrix} \phi_{I,x} & 0 \\ 0 & \phi_{I,y} \\ \phi_{I,y} & \phi_{I,x} \end{bmatrix}$$

$$f_I = \int_{\Omega} \Phi_I^T b d\Omega + \int_{\Gamma_t} \Phi_I^T \bar{t} d\Gamma$$

$$q_I = - \int_{\Gamma_u} N_I^T \bar{u} d\Gamma \quad N_I = \begin{bmatrix} N_I & 0 \\ 0 & N_I \end{bmatrix}$$

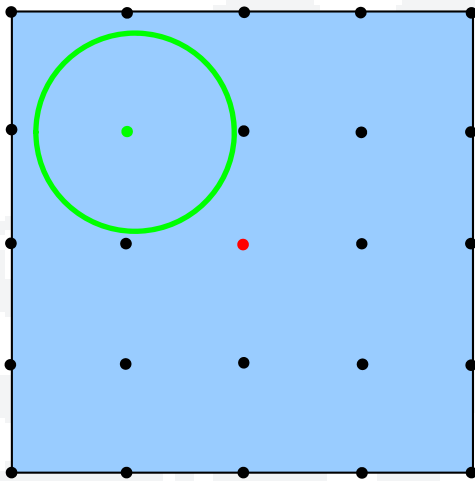
EFG weak Formulation:

1. It is evident that there is a need to integrate over the problem domain and the curve integrations for both natural and essential boundaries.
2. These integrations have to be carried out via numerical quadrature techniques. The Gauss quadrature scheme is the most often used.
3. In using the numerical quadrature scheme, a mesh of cells; called a background mesh is required for the integration.

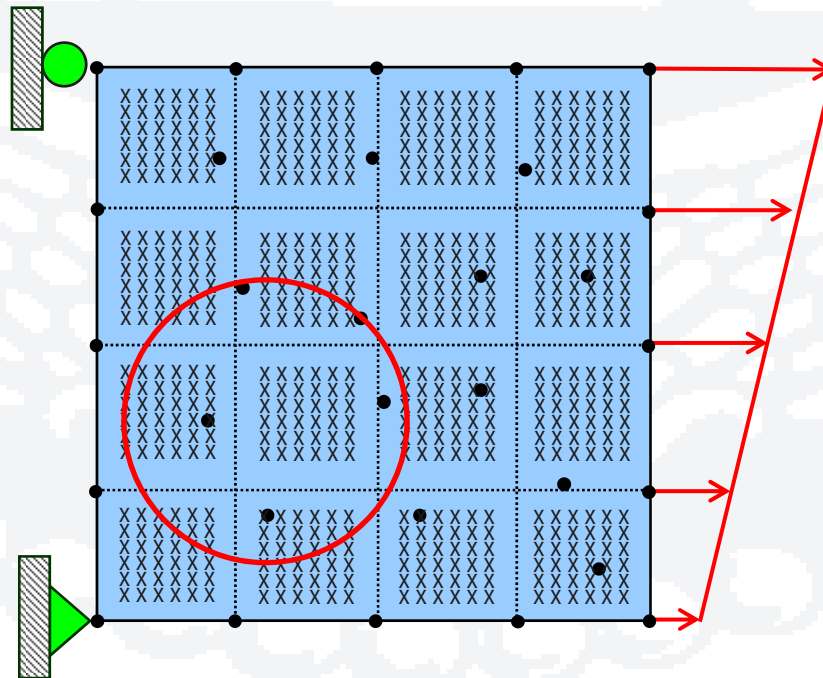
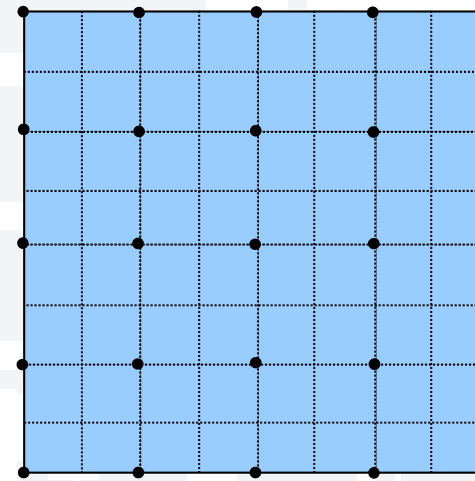
EFG weak Formulation:

4. The solution efficiency of the enlarged system can be drastically reduced, because:
 - K can be much larger than the FE stiffness matrix.
 - Presence of G .
 - The matrix of the enlarged system is:
 1. Symmetric
 2. Not positive definite.

EFG background integration:



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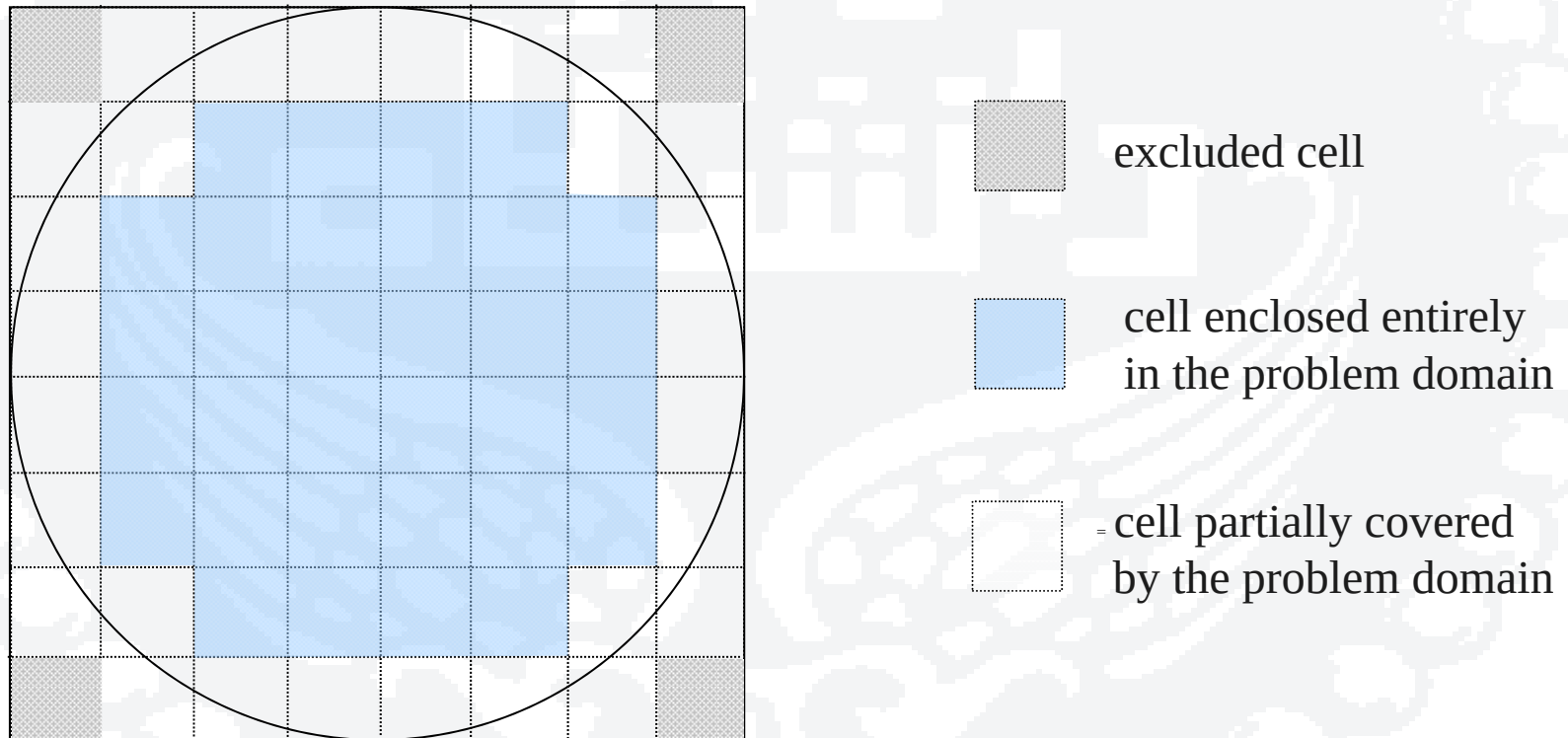


EFG background integration:

For 2D stress analysis:

1. The minimum number of integration points must be greater than two thirds of the total number of the unfixed field nodes.
 1. This is only the minimum requirement to ensure a non-singular system matrix.
 2. It does not guarantee the accuracy of the solution.
2. The necessary ratio of the integration points to the field nodes is estimated to be around 3 to 9.
3. The displacement field can be obtained rather accurately with a course background mesh of integration cells.
4. A fine background mesh, however, is necessary for computing the stress field.

EFG background integration:



EFG Flowchart:

1. Geometry generation.
2. Node and background mesh generation.
3. For all the cells in the background mesh.
 - a) For all the quadrature points x_Q in a cell.
 - i. Search in the current cell and its surrounding cells of integration for all the nodes that fall in the support domain of x_Q .
 - ii. MLS shape function creation based on selected nodes.
 - iii. Calculate nodal matrices.
 - iv. Assemble the nodal matrix into the global matrix.
 - b) ENDDO

EFG Flowchart:(cont.)

4. EDNDO
5. Solve the system equation for nodal parameters of displacements.
6. Calculate displacements and the derivatives using MLS shape function.
7. Calculate strain and stress.

EFG with Penalty Method:

- This is an alternative EFG method which uses the penalty method for the imposition of boundary conditions.
 1. The method produces equations of the same dimension as the conventional FEM for the same number of nodes.
 2. The modified stiffness matrix is still positive definite.
 3. The modified stiffness matrix is banded.
 4. The treatment of boundary conditions is as simple as it is in conventional FEM.
- EFG with penalty method is applicable in handling problems of composite materials, where the continuity conditions between different types of materials need to be enforced.

EFG with Penalty Method:

- The method uses a penalty factor to penalize the difference between the displacement of MLS approximation and the prescribed displacement on the essential boundary.
- The constrained Galerkin weak form using the penalty method can be written as:

$$\int_{\Omega} \delta(Lu)^T C(Lu) d\Omega - \int_{\Omega} \delta u^T b d\Omega - \int_{\Gamma_t} \delta u^T \bar{t} d\Gamma \\ - \delta \int_{\Gamma_u} \frac{1}{2} (u - \bar{u})^T \alpha (u - \bar{u}) d\Gamma = 0$$

EFG with Penalty Method:

$$\int_{\Omega} \delta(Lu)^T C(Lu) d\Omega - \int_{\Omega} \delta u^T b d\Omega - \int_{\Gamma_t} \delta u^T \bar{t} d\Gamma - \delta \int_{\Gamma_u} \frac{1}{2} (u - \bar{u})^T \alpha (u - \bar{u}) d\Gamma = 0$$

$$\alpha = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_k \end{bmatrix}$$

- k is the dimension of the problem (2 for 2D and 3 for 3D)
- α is a diagonal matrix of penalty factors.
- The penalty factors α_i can be a function of coordinates.
- They can be different from node to node.
- In practice, an identical large positive constant number is used for all nodes.

EFG with Penalty Method:

$$[K + K^\alpha]U = F + F^\alpha$$

$$F = \bigcap_{I=1}^{n_t} f_I$$

$$f_I = \int_{\Omega} \Phi_I^T b d\Omega + \int_{\Gamma_t} \Phi_I^T \bar{t} d\Gamma$$

$$K = \bigcap_{I,J=1}^{n_t} K_{IJ}$$

$$K_{IJ} = \int_{\Omega} B_I^T C B_J d\Omega$$

$$F^\alpha = \bigcap_{I=1}^{n_t} f_I^\alpha$$

$$f_I^\alpha = \int_{\Gamma_u} \Phi_I^T \alpha \bar{u} d\Gamma$$

$$K^\alpha = \bigcap_{I,J=1}^{n_t} K_{IJ}^\alpha$$

$$K_{IJ}^\alpha = \int_{\Gamma_u} \Phi_I^T \alpha \Phi_J d\Gamma$$

EFG with Penalty Method:

1. The integrations are performed along the essential boundary.
2. The matrix K^α will have entries only for the nodes near the essential boundaries Γ_u , which are covered by the support domains of all the quadrature points on Γ_u .

EFG with Penalty Method:

Advantages:

1. The dimension of the system matrix is preserved.
2. The system matrix is positive definite, as long as the penalty factors are positive.
3. The system matrix is symmetric.
4. The bandness of the system matrix is preserved.
5. The method is 2-10 times faster than the Lagrange method (for simple small 2D problems).

EFG with Penalty Method:

Disadvantages:

1. It is necessary to choose penalty factors to be universally applicable for all kinds of problems.
2. Too large penalty factors often result in numerical problems.
3. The results are in general less accurate in comparison to the results obtained with Lagrange multipliers.
4. Essential boundary conditions are only approximately imposed.

EFG-Penalty for Material Discontinuity Conditions:

- The domain of a problem can be composed of subdomains of different materials.
- No special treatment is required in FEM.
- Special treatments are required in meshless EFG method.

1. Cordes/Moran (1996)

A special boundary condition using Lagrange multipliers

2. Krongauz/Belytschko (1998)

Adding special shape functions that contain discontinuities in derivatives

3. Liu/Yang (1999)

Penalty method to enforce special boundary conditions.

EFG-Penalty for Material Discontinuity Conditions:



EFG-Penalty for Material Discontinuity Conditions:

1. For an inhomogeneous body, which consists of two homogeneous bodies, an interface is defined by a set of nodes that belong to both materials.
2. A non-penetration (impenetrability) rule is imposed for the influence domains of the nodes:
 - Points contained in material 1 can only be influenced by the nodes in material 1 plus the interface nodes. (and similarly for points in material 2)
 - This rule confines the influence domain of a node within the subdomain of the material of the node.

EFG-Penalty for Material Discontinuity Conditions:

- For the homogeneous case:
 - Point a is contained in the influence domains of 4 and 5.
 - Point b is contained in the influence domains of 3 and 5.
 - Point c is contained in the influence domains of 1 and 2.
- For the inhomogeneous case:
 - Point a is contained in the influence domains of 4 and 5.
 - Point b is contained in the influence domains of 3 and 5.
 - Point c is contained in the influence domains of 1.

EFG-Penalty for Material Discontinuity Conditions:

- The connection of the subdomains is achieved by enforcing conditions of continuity on the interface of the two material Γ_s :

$$\bar{u}^+ = \bar{u}^-$$

$\bar{u}^+, \bar{u}^-$ Correspond to the displacement in the two materials on the interface of Ω^+ and Ω^- , respectively.

EFG-Penalty for Material Discontinuity Conditions:

- The constrained Galerkin weak form using the penalty method:

$$\begin{aligned} & \int_{\Omega} \delta(Lu)^T C(Lu) d\Omega - \int_{\Omega} \delta u^T b d\Omega - \int_{\Gamma_t} \delta u^T \bar{t} d\Gamma \\ & - \delta \int_{\Gamma_u} \frac{1}{2} (u - \bar{u})^T \alpha (u - \bar{u}) d\Gamma \\ & - \delta \int_{\Gamma_s} \frac{1}{2} (\bar{u}^+ - \bar{u}^-)^T \beta (\bar{u}^+ - \bar{u}^-) d\Gamma = 0 \end{aligned}$$

- β is a diagonal matrix of penalty factors that have the same form of α , but with different values.

EFG-Penalty for Material Discontinuity Conditions:

- The approximate values of $\bar{u}^{+h}$ and $\bar{u}^{-h}$ are expressed using MLS approximations:

$$\bar{u}^{+h}(x) = \sum_{I=1}^n \Phi_I^+(x) u_I^+$$
$$\bar{u}^{-h}(x) = \sum_{I=1}^n \Phi_I^-(x) u_I^-$$

Φ_I^+ is the matrix of shape functions using the nodes that have influence on $\mathbf{x}$ in Ω^+

Φ_I^- is the matrix of shape functions using the nodes that have influence on $\mathbf{x}$ in Ω^-

n^+, n^- are the number of nodes that have influence on $\mathbf{x}$ in Ω^+ and Ω^- .

EFG-Penalty for Material Discontinuity Conditions:

- The final set of discretized system of equations can be written as:

$$[K + K^{\alpha} + K^{\beta}]U = F + F^{\alpha}$$

$$K^{\beta} = \bigcap_{I,J=1}^{n_{\beta}} K_{IJ}^{\beta}$$

$$K_{IJ}^{\beta} = \int_{\Gamma_s} [\Phi_I^+ - \Phi_I^-]^T \beta [\Phi_I^+ - \Phi_I^-] d\Gamma$$

- K^{β} is the stiffness matrix for connecting the two different materials.
- The integration for K^{β} is performed along the interfaces of materials.
- K^{β} will have entries for the nodes near (and not only on) the interfaces, which have influence on the quadrature points on the interface of different materials.

